Math 2312 Notes

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1. Derivative at a point

We (as in you and I) want to first define the derivative of a function at a point. This yields the slope of the tangent line at the point $(a, f(a))$, if the limit exists.

Definition 1 (Derivative at a point). *The derivative of a function $f(x)$ at a value $x = a$ is the following limit, when the limit exists.*

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

When this limit exists, we denote it by $f'(a)$.

There is an equivalent definition for the derivative of a function at a point. When we say a definition is equivalent to another, we mean that they are saying the same thing, just differently. We have become familiar with the idea that two things can look different, but be mathematically the same. This is what 'equivalency' means.

Definition 2 (Equivalent definition of Derivative at a point). *The derivative of a function $f(x)$ at a value $x = a$ is the following limit, when the limit exists.*

$$\lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

When the limit exists, we denote it by $f'(a)$.

Geometrically, the derivative of a function at a point is an algebraic concept. It's geometric interpretation is the slope of a tangent line at the point $(a, f(a))$.

2. Derivative as a function

Definition 3 (Derivative as a function). *We define a function $g(x)$ as follows. Let $f(x)$ be a function. Then the following function is called the derivative of $f(x)$ and, when defined, is a function*

Theorem 1 (Derivative of a constant function). *If $f(x) = c$ is a constant function, then the derivative of $f(x)$ exists and is $f'(x) = 0$, for every x on the real line.*

When we want to prove anything, we go back to the definitions of the terms in the statement. This is where we start.

Proof. Consider a function $f(x) = c$. We want to use the definition of the derivative to show that $f'(x) = 0$ for every x. This means we want to calculate the following limit:

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Since $f(x) = c$ for any x, $f(x+h) = c$, since "$x+h$" is any value on the real line, too, right?! We then use this to calculate the limit:

$$\begin{aligned}
\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \to 0} \frac{c - c}{h} \\
&= \lim_{h \to 0} \frac{0}{h} \\
&= \lim_{h \to 0} 0 \\
&= 0
\end{aligned}$$

The limit exists for every x, so the derivative as a function of x exists where defined, which, in this case, is every x. □

Theorem 2 (derivative of x). *The derivative (as a function of x) of $f(x) = x$ is $f'(x) = 1$. In other words, it is a constant function.*

Proof. We proceed just as we did in the proof that the derivative of a constant function was zero:

$$\begin{aligned}
&(1) & \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \to 0} \frac{x + h - x}{h} \\
&(2) & &= \lim_{h \to 0} \frac{h}{h} \\
&(3) & &= \lim_{h \to 0} 1 \\
&(4) & &= 1
\end{aligned}$$

The limit is a function of x and defines a function $g(x)$ called the derivative of $f(x)$, which we denote by $f'(x)$ to remind us of the notation we used for the derivative of a function at $x = a$ (or geometrically, the slope of the tangent line at $(a, f(a))$. Since we have shown what we wanted to show, we are done. That is, the derivative of $f(x) = x$ is indeed a constant function $f'(x) = 1$. □