Math 2312 Notes
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1. Limits

We should be over our intuitive understanding of the word "limit." It is not necessarily something we need worry about obtaining, but something we know that we can get arbitrarily close to. Just like how the sequence

$$\{-1, 1, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{3}, \frac{1}{3}, \cdots\}$$

doesn't run up against any *limit* in the sense you have envisioned based on your everyday use of the word, limits, in general, are not something one necessarily climbs towards or constitutes some barrier. But, we can say for certain that if the "limiting value" of some sequence is X, then the numbers in that sequence continue to get closer and closer to X as one lists more and more numbers in the sequence.

When one calculates the limit of a function, one is calculating the *eventual* behavior of that function at a number that may or may not be in the domain of the function. The best way to investigate the possible limit of a function is to make a table of values.

Example 1. Consider the function $f(x) = x^2$. We want to know what value $f(x)$ gets closer to as x gets closer and closer to 1. We make a table to investigate the behavior of the function around $x = 1$.

x	y
0.5	0.25
0.75	0.5625
0.95	0.9025
0.99	0.9801
0.999	0.998001
1.5	2.25
1.25	1.5625
1.05	1.1025
1.01	1.0201
1.001	1.002001

From this table, it would appear that $f(x)$ will get closer to 1 as x gets closer to 1. Likewise, we could also built a table to convince ourselves that as x gets closer and closer to, say 4, that $f(x)$ gets closer and closer to 16. Notice the pattern?

For now (that is, until we can provide you with the rigorous definition of limit), calculating a limit of a function is all about calculating the eventual behavior based on observations. But your eyes can be deceived. The book provides a great example of this.

Example 2. Consider the function $f(x) = x^3 + \frac{\cos 5x}{10{,}000}$.

x	y
1.0	1.000028
0.5	0.124920
0.1	0.001088
.05	0.000222
0.01	0.000101

From this table, you get the impression that the limit is going to be zero, right? As the x values become smaller and smaller, the output of the function gets smaller and smaller. It seems like a reasonable conclusion, but is it? If we input smaller values into our function, what is the output? How small do we need to go to be *sure* our intuition is not leading us astray? This is indeed the problem with an *intuitive definition of limit.*

In fact, the limit will be exactly 0.0001 and not zero. We will see later why this is the case, but you can convince yourself of this by inputting 0.001 and 0.0001 into the function. The output will be so close to 0.0001 each time that it is apparent you will not be able to get an output less than that.

Another example you should read in the text is $f(x) = \sin(\pi/x)$. As x gets closer and closer to zero, is it clear that $f(x)$ will get closer and closer to anything? No, it is not clear. Time permitting, we'll make our way through this example in class.

1.1. **Left and right side limits.** We've made tables that investigate the behavior of a function near values on the *left and right* of a point on the x-axis (typically, the domain of the function. When we only examine the behavior of a function near a point on the x-axis, but only on one side of it, we are in fact talking about the *left* or *right side limit* of the function. When might these be different?

Example 3. Consider the function $f(x) = \frac{x}{|x|}$. This is another way of writing what the book calls the "Heavyside Function." It's named after a person named Heavyside. To the best of my knowledge, they were not overweight or oddly proportioned.

We want to examine the behavior of $f(x)$ around $x = 0$. Well, plugging in any value to the left of zero gives us $f(\text{any value to left of zero}) = -1$. Therefore, we say the left side limit of $f(x)$ as x approaches zero is -1. But the right side limit is then $+1$, since $|x| = x$ when $x > 0$. This gives us a situation where the left side limit and the right side limit are not the same.

When the left side limit and the right side limit are not the same or are not finite, we say the limit of the function as x approaches a value in question does not exist. When the left and right side limits are equal and finite, then the limit does exist.

1.2. **Infinite limits.** An infinite limit is a special case of the limit not existing. We've all seen vertical asymptotes in precalculus and before. We now give a *limit definition* of vertical asymptote, which will be in terms of what we now call an infinite limit.

Consider the function $f(x) = \frac{1}{x-1}$. We can see that this function is not defined at the number $x = 1$; plugging it into the function makes bad things happen. But it doesn't stop us from investigating the behavior of the function near $x = 1$, does it? Of course not! But we can quickly see from our calculators that inputs that are closer and closer to 1 from the right result in outputs that are larger and larger and positive. Likewise, plugging in values that are closer and closer to 1 from the left result in outputs that are larger and larger and **negative**. Therefore, the left side limit is $-\infty$ and the right side limit is $+\infty$. The *infinite limit does not exist* since the left and right side infinite limits are not equal. But it is possible for the *infinite limit* to exist (but this wont mean that the limit will exist). Consider a similar function $g(x) = \frac{1}{(x-1)^2}$. The infinite limit of $g(x)$ as x gets closer and closer to 1 from the left or right will be $+\infty$. Therefore, the *infinite limit* exists, but the limit does not (since the limit would not be finite). Remember, an infinite limit is a special case of a limit not existing, but in a way that is still meaningful.

1.2.1. *Vertical asymptotes.* Consequently, as promised, we have a new definition for vertical asymptote. If the left or right side infinite limit is ever $+\infty$ or $-\infty$, then we say a vertical asymptote occurs at the x value one was approaching.