Math 2312 Notes
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1. Limits at infinity

We've spent a fair amount of time on investigating the behavior of a function $f(x)$ at x values near some value $x = a$ (i.e., some value on the x-axis but not necessarily in the domain of the function).

There are other aspects of a graph that can be quite telling and one of those aspects is the behavior of the function $f(x)$ as x becomes really large and positive or really large and negative. This behavior is called the "limit at infinity." We denote the limit at $+\infty$ as $\lim_{x\to\infty} f(x)$ and the limit at $-\infty$ as $\lim_{x\to-\infty} f(x)$.

The best example you can think of is the exponential function $f(x) = e^x$. As x becomes arbitrarily large and positive, what behavior does e^x exhibit? The values of e^x become arbitrarily large, as well. As x becomes arbitrarily large and negative, what behavior does e^x exhibit? The function e^x becomes arbitrarily small.

Rational functions may exhibit particularly interesting behavior for arbitrarily large values.

Example 1. Consider the function

$$f(x) = \frac{3x - 1}{2x + 5}$$

What does the graph look like? What happens for arbitrarily large values of x, be they positive or negative? We know right away from our experience that $x = -\frac{5}{2}$ is not in the domain of the function. Away from that particular value, the function will be defined (i.e., make sense). But what is the eventual behavior of the function as x gets arbitrarily large? Eventually, the function will settle on a particular value. The idea of eventually means "as x goes to ∞ or $-\infty$." It appears that the function is settling on the value $y = \frac{3}{2}$. But how can we be sure?

Most of the limit laws we saw in the previous sections can be adapted to make sense of $\lim_{x\to\infty} f(x)$ or $\lim_{x\to-\infty} f(x)$. But how will they help us? What happens when the numerator and denominator in $f(x)$ get arbitrarily large? We end up with, effectively, ∞/∞. But what is this? ∞ is not a number so we cannot immediately say that $\infty/\infty = 1$. This is not necessarily the case. In fact, saying this would trivialize most of what we are about to do.

What we can do is factor out from the numerator and the denominator the same term, x. This gives us:

$$f(x) = \frac{x\left(3 - \frac{1}{x}\right)}{x\left(2 + \frac{5}{x}\right)}.$$

Now, $x/x = 1$. Examining the limit $\lim_{x\to\infty} \frac{3-1/x}{2+5/x}$, we see that as x becomes arbitrarily large, $1/x$ and $5/x$ become arbitrarily small. Therefore the limit is $3/2$. And, as our graph indicates (the one you plotted), the graph appears to be getting closer and closer to the line $y = 3/2$.

One very popular misconception from Precalculus is that a horizontal asymptote is a barrier to the graph of the function. That is, if the graph of a function is getting arbitrarily close to a horizontal line and does not cross it, then such a line is called a horizontal asymptote of the function. This is unnecessary.

Definition 1 (Horizontal Asymptotes). *If* $\lim_{x\to\infty} f(x)$ *exists (i.e., is finite), then such a value constitutes a constant function and a horizontal asymptote for the function. Likewise, if* $\lim_{x\to-\infty} f(x)$ *exists, then such a value constitutes a constant function and a horizontal asymptote for the function.*

Example 2. $\lim_{x\to\infty} \frac{\sin x}{x} = 0$ and $y = 0$ is a horizontal asymptote for $f(x) = \frac{\sin x}{x}$. However, the graph of $\sin xx$ intersects infinitely often with the graph of $y = 0$. This probably runs counter to your intuition of what a horizontal asymptote is.

Let's ask ourselves a question. Suppose we have a function $f(x)$ where $\lim_{x\to\infty} f(x) = +\infty$ (it could just as well equal $-\infty$, but I'll leave it to you to see that it doesn't change much). In addition to knowing the limit of $f(x)$ is an infinite limit (i.e., does not exist, but there is still something discernible), suppose we also have a function $g(x)$ where $\lim_{x\to\infty} g(x)$ exists (i.e., is finite). Then, we have that $\lim_{x\to\infty} f(x)g(x) = +\infty$.

Example 3. Consider $f(x) = \dfrac{3x^2 - 2x + 1}{x - 1}$. This function is not defined for $x = 1$. But this does not prevent us from examining the behavior of the function as x gets large (be it negative or positive). That is, we can still attempt to compute the limit of the function as x goes to $\pm\infty$. If we naïvely attempt to apply the quotient law, we run into a problem:

$$\begin{aligned} \lim_{x\to\infty} \frac{3x^2 - 2x + 1}{x - 1} &= \frac{\lim_{x\to\infty} 3x^2 - 2x + 1}{\lim_{x\to\infty} x - 1} \\ &= \frac{\infty}{\infty} \\ &\neq 1. \end{aligned} \tag{1}$$

We see that the naïve application of the quotient law leads us astray. This is not helpful. We must manipulate the function using algebra to get it to look different, but still be mathematically the same. We do the following. I suggest you write out the calculation as you're reading it. It will make it stick that much better.

$$\begin{aligned} \lim_{x\to\infty} \frac{3x^2-2x+1}{x-1} &= \lim_{x\to\infty} \frac{x^2(3-2/x+1/x^2)}{x(1-1/x)} \\ &= \lim_{x\to\infty} \frac{x(3-2/x+1/x^2)}{1-1/x}. \end{aligned} \tag{2}$$

We are almost ready to determine whether or not there is any discernible behavior (that is, calculate the limit). Let's break down our function as we now have it written in the last line of the previous calculation: for arbitrarily large (positive or negative) values of x, we have that

$$\frac{3x^2-2x+1}{x-1} = \frac{x(3-2/x+1/x^2)}{1-1/x} \tag{3}$$

In the paragraph preceding "Example 3," we make a comment about a product of functions, where one grows arbitrarily large, but the other function's limit is finite *and* nonzero. We have exactly that situation now. That is, by the quotient law

$$\lim_{x\to\infty} \frac{3-2/x+1/x^2}{1-1/x} = \frac{3}{2} \tag{4}$$

and

$$\lim_{x\to\infty} x = \infty. \tag{5}$$

Therefore, $\lim_{x\to\infty} f(x) = \infty$.

The book may or may not list a table of rules for dealing with rational function in the context of "limits at infinity." If it does not, see if you can construct a list of rules for when the degree of the polynomial in the numerator of the rational function is 1) larger than the degree of the polynomial in the denominator, 2) equal and 3) smaller.