Math 2312 Notes
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1. Examples using the Squeeze Theorem

Example 1. It is shown in the notes that $\lim_{x\to 0} x\sin(1/x) = 0$. But what about $\lim_{x\to 0} x^2\sin(1/x^2)$? Your first instinct may be to use the power law for limits. But this would be a bad instinct since $\sin(1/x^2) \neq \sin^2(1/x)$. So we cannot willy-nilly apply the power law since we cannot show that it applies to the given function. But we can follow a similar procedure as we did in the example from the notes.

We know that the function $\sin(1/x^2)$ is bounded above by the line $y = 1$ and blow by the line $y = -1$. That is,

$$-1 \leq \sin(1/x^2) \leq +1.$$

Then,

$$-x^2 \leq x^2\sin(1/x^2) \leq x^2.$$

This inequality is true for all $x \neq 0$. But that is all we care about when calculating the limit of $x^2\sin(1/x^2)$ as x goes to zero using the Squeeze Theorem. Since $\lim_{x\to 0} -x^2 = \lim_{x\to 0} x^2 = 0$, the Squeeze Theorem tells us that $\lim_{x\to 0} x^2\sin(1/x^2) = 0$.

Example 2. If possible, compute $\lim_{x\to 0} x^2 e^{\sin 1/x}$. How should we proceed? Should we attempt to use the product law for limits? If we do, we must show that $\lim_{x\to 0} e^{\sin(1/x)}$ exists. But does it? If it does not exist, then we cannot use the product law for limits. The truth is, the $\lim_{x\to 0} e^{\sin(1/x)}$ does not exist. Recall that as $x \to 0$, the behavior of $\sin(1/x)$ is not definitive. One could argue that the function wants to settle on the value 1 and -1 at the same time, meaning that the limit does not exist. So if the behavior of $e^{\sin(1/x)}$ as $x \to 0$ can arguably be e^{-1} and e^1 simultaneously, then the limit does not exist. Therefore, we cannot use the product law for limits.

We turn to the Squeeze Theorem for help. As in the previous example, we start with $-1 \leq \sin(1/x) \leq 1$. Then,

$$e^{-1} \leq e^{\sin(1/x)} \leq e^1.$$

Now, we multiply through by x^2. Since this is always a positive quantity, the inequalities will not switch direction. Therefore, except for when $x = 0$, we have that

$$x^2e^{-1} \leq x^2e^{\sin(1/x)} \leq x^2e.$$

Now, $0 = \lim_{x\to 0} x^2 e^{-1} = \lim_{x\to 0} x^2 e$ (remember: e is a nonzero constant, so you can use the constant multiple law). Therefore, by the Squeeze Theorem, the limit is zero.

Example 3 (Extra credit: 0.2% of your grade)**.** If possible, compute $\lim_{x\to 0} \frac{\sin x}{x}$. First, notice how this is different from any of the examples we have done. The function $\sin x$ is now defined at zero, but $\lim_{x\to 0} = 0$, so we cannot use the quotient law for limits.

What do we do? Don't look on the internet. I'll know if you simply write down the answer without understanding the solution. Hint: justify the following inequality,

$$\cos x \leq \frac{\sin x}{x} \leq 1$$

You may use the fact that $\lim_{x\to 0} \cos x = 1$.

Bring your solution to office hours and explain to me the solution. Must be done by September 20, 2017, 5:00 PM.