Math 2312 Notes

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1. Derivatives of trigonometric functions

Theorem 1 (Derivative of $\sin x$ and $\cos x$)**.** *The derivative (as a function of x) of $\sin x$ is $\cos x$. The derivative of $\cos x$ is $-\sin x$.*

Proof. Lots of trig, the squeeze theorem, and much of what you would have done on the extra credit assignment. It's not hard to understand, it's just very involved. □

This is a good time to introduce the notation for *the derivative of the derivative* or "the second derivative." As you might expect, we repeat the use of the prime notation. If f is differentiable (or, in other words, its derivative exists as a function of x), then $f'(x)$ is the derivative. [1] But what if $f(x)$ was already the derivative of some function $g(x)$? That is, $f(x) = g'(x)$. Then $f'(x) = (g'(x))'$. We write this more compactly as $g''(x)$. Let's now look at the second derivative of $\sin x$. We now know that the first derivative of $\sin x$ is $\cos x$ and that the first derivative of $\cos x$ is $-\sin x$. So what is the second derivative of $\sin x$? It is the derivative of $\cos x$, which is $-\sin x$. We have the following:

$$\begin{aligned}(\sin x)'''' &= (\cos x)''' \\ &= (-\sin x)'' \\ &= (-\cos x)' \\ &= \sin x\end{aligned}$$

That is, the fourth derivative of $\sin x$ is equal to $\sin x$. We use this to our advantage many times in Calculus II. A similar calculation shows us that the fourth derivate of $\cos x$ is equal to $\cos x$. The cyclic nature of the the nth derivative is no coincidence. It is tied to the fact that sin and cos are periodic functions.

We now want to talk about the product rule and quotient rule. The latter will be useful when trying to calculate the derivative of $\tan x$, since this is expressed as a quotient of two functions, namely $\sin x$ and $\cos x$.

Theorem 2 (Quotient rule for derivatives)**.** *If $f(x)$ and $g(x)$ are differentiable functions, then the derivative of $f(x)/g(x)$ is:*

[1]**In Calculus III, you will learn more about the notion of differentiability. For now, we play a little fast and loose with the definition**.

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - g'(x)f(x)}{[g(x)]^2} \tag{1}$$

So how can we use this theorem to help us calculate the derivative of $\tan x$? Recall that $\tan x = \sin x / \cos x$. Tangent is the quotient of two differentiable functions. Hence, we can use the quotient rule to calculate the derivative of $\tan x$.

Theorem 3 (Derivative of $\tan x$)**.** *The derivative of* $\tan x$ *as a function of* x *is* $\sec^2 x$*.*

Proof.

$$(\tan x)' = \frac{(\sin x)'\cos x - (\cos x)'\sin x}{\cos^2 x} \tag{2}$$

$$= \frac{\cos x \cos x - -\sin x \sin x}{\cos^x} \tag{3}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \tag{4}$$

$$= \frac{1}{\cos^2 x} \tag{5}$$

$$= \sec^2 x \tag{6}$$

□

See if you can write down the justification for each line in the calculation.

The derivatives of $\sec x$, $\csc x$ and $\cot x$ can be similarly calculated from the quotient rule for derivatives.

Now, what some of you will inadvertently do on the exam is say that the derivative of the quotient of functions is the quotient of their derivatives. This is wrong, as you can clearly see from the theorem.

Let's move on to the product rule.

Theorem 4 (Product rule for derivatives)**.** *If* $f(x)$ *and* $g(x)$ *are differentiable functions, then* $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$*.*

Again, what many of you will do on the exam and in your own calculations is incorrectly say that the derivative of the product of two differentiable functions is the product of their respective derivatives. This is incorrect, as the theorem states.

The product rule for derivatives allows us to compute the derivatives of much more complicated functions.

Example 1. Compute the derivative of $x \sin x$. To do so, we use the product rule:

$$(x \sin x)' = (x)' \sin x + x(\sin x)' \tag{7}$$
$$= 1 \sin x + x \cos x \tag{8}$$
$$= \sin x + x \cos x \tag{9}$$

Example 2. Use the quotient rule for derivatives and product rule for derivatives to compute the derivate of $f(x) = x \sin x/(x^2 + 1)$.