Math 2312 Notes

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1. Chain rule

Recall the notation presented in the miscellaneous notes.

Notation 1 ($\frac{d}{dx}$ notation for derivatives)**.** *The derivative $f'(x)$ of a differentiable function can be expressed as $\frac{d}{dx}f(x)$. More succinctly, when $y = f(x)$, we write $\frac{dy}{dx}$. This is called the Leibniz notation.*

- nth derivatives in the Leibniz notation: $\frac{d^n}{dx^n}f(x)$. More succinctly, when $y = f(x)$, we write $\frac{d^n y}{dx^n}$.

We have discussed how to compute the derivative of a sum of two differentiable functions, the product of two differentiable functions and the quotient of two differentiable functions. We've also discussed how to compute the derivative of a constant function and the derivative of a constant function times a differentiable function. We have not discussed how to compute the derivative of a function of the form, for example, $\sin(x^2 + 1)$. None of the rules we have at our disposal will be of immediate use in calculating the derivative of this function.

Let's look at an easier example before we present the statement on the chain rule.

Example 1. What is the derivative of $h(x) = (x^2 - 2x + 1)^3$? We could first multiply $x^2 - 2x + 1$ by $x^2 - 2x + 1$ and then again once more to determine the expanded form of the expression. The result would be a six-degree polynomial:

$$h(x) = x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1$$

We know how to compute the derivative of this function. We just use the power rule, constant multiple rule and sum rule for derivatives and the fact that the derivative of a constant is zero:

$$h'(x) = 6x^5 - 30x^4 + 60x^3 - 60x^2 + 30x - 6$$

But what if we didn't want to go through all of that trouble? What if we wanted an immediate way of calculating the derivative of the polynomial without first expanding the expression?

Let's suppose instead of the third power, we had the following: $(x^2 - 2x + 1)^{2017}$.

Do you want to compute that power? Do you want to bother finding a pattern in the derivative, let alone the expansion? No, no you don't, because you're S-M-R-T, SMART! This is why the chain rule is so useful. It makes easier work of this. But what does it say?

Loosely speaking, the chain rule says that the derivative of a composition of differentiable functions exists: $(f \circ g(x))' = f'(g(x)) \cdot g'(x)$. Do not confuse this with the product rule for derivatives.

Let's use this to find the derivative of $h(x) = (x^2 - 2x + 1)^3$. Notice that this function can be written as the composition of two functions $f(x) = x^3$ and $g(x) = x^2 - 2x + 1$. Remember, x is just a variable that can stand in for almost anything. When we compose to functions, $f \circ g(x)$ means that we are evaluating f at the output of $g(x)$. We are effectively considering a narrower view of the domain of f when we do this (possibly narrower, we could still not be leaving out any values. For instance, $f(x) = x^{1/3}$ and $g(x) = x^2 + 1$. The domain of $f(x)$ is all real numbers. The range of $g(x)$ is all positive real numbers. When we compose $f(x)$ with $g(x)$, we are effectively evaluating $f(x)$ only at positive real numbers. The domain of $f(x)$ didn't really change, just the values we're considering plugging into $f(x)$. The domain of $f \circ g(x)$, however, is all real numbers. $g(x)$ is still a function of all real numbers. If $g(x) = \frac{1}{x^2-1}$, then the range would not include zero, so $f \circ g(x)$ would never be zero (try to find a solution to the equation $f \circ g(x) = 0$ and you'll see you cannot). The domain of $g(x)$ does not include ± 1, so the values we plug into $f(x)$ are not all of the values that we would *usually be able to plug into* $f(x)$.

Now, you've probably read the last paragraph and got overwhelmed. That's natural. You should feel a bit overwhelmed by it. So what do you do? Break it apart line by line and write it out and understand every sentence.

There are two ingredients we need to compute the derivative of $h(x) = (x^2 - 2x + 1)^3$. We need the derivative of $f(x)$ evaluated at $g(x)$ and the derivative of $g(x)$. The derivative of $f(x)$ is $f'(x) = 3x^2$. The derivative of $f(x)$ **evaluated at** $g(x)$ is $f'(g(x)) = 3[g(x)]^2 = 3(x^2 - 2x + 1)^2$. The derivative of $g(x) = x^2 - 2x + 1$ is $g'(x) = 2x - 2$. Then, by the chain rule, $f \circ g(x)$ is $f'(g(x))g'(x) = 3(x^2 - 2x + 1)^2(2x - 2)$. If you expand this out, you will see that $(f \circ g(x))' = h'(x)$ from above.

We can now use the chain rule to compute the derivative of $h(x) = \sin(x^2 + 1)$. Identify the two ingredients: $f(x) = \sin x$ and $g(x) = x^2 + 1$. Since $f'(g(x)) = \cos(x^2 + 1)$ and $g'(x) = 2x$, we have that $(f \circ g(x))' = 2x \cos(x^2 + 1)$.