Math 2312 Notes
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1. Continuity

Definition 1 (Continuous at a point). *A function $f(x)$ is continuous at a point $x = a$ means that* $\lim_{x\to a} f(x) = f(a)$.

Implicit in this definition are three consequences:

(1) The value a is in the domain of $f(x)$.
(2) The limit exists
(3) The limit L is $f(a)$.

There is an alternate definition of continuity that builds on the existing rigorous definition of limit:

Definition 2 ($\varepsilon - \delta$ definition of continuity). *A function $f(x)$ is continuous at $x = a$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $|x - a| < \delta$, it is the case that $|f(x) - f(a)| < \varepsilon$.*

This is building on everything we have seen so far. We can show a function is continuous using either definition, but, at the end of the day, we're really using the 2nd definition if we're doing it right.

Ask yourself: how do we show L is the limit? Using the limit laws or the $\varepsilon - \delta$ definition of limit. Once we've done that, we must verify that a is in the domain of the function and $f(a) = L$. Or for more basic functions, we have the limit laws and the fact that $\lim_{x\to a} x = a$ and $\lim_{x\to a} c = c$. Note, these last two "facts" can be proven using the $\varepsilon - \delta$ laws.

Because we are building on everything we have seen so far, the limit laws for when $x \to a$ all hold when discussing continuous functions. (The power law and the nth-root law already say when such functions are continuous. Notice how $L = f(a)$ in those statements.)

One important fact is that if $f(x)$ and $g(x)$ are continuous functions and the range of $g(x)$ is in the domain of $f(x)$, then $f \circ g(x)$ is a continuous function. But this is not to imply that when the composition of two functions is a continuous function that each individual function must be continuous. The easy example illustrating this is a constant function $f(x)$ and any discontinuous function $g(x)$ defined for all real numbers (think of the greatest integer function). Composing f with g results in a constant function, which is clearly continuous (see if you can prove this using the $\varepsilon - \delta$ definition of continuity).

When a function is continuous at every point in an open interval I, we say the function is continuous over the open interval.

Definition 3 (Continuous over an open interval). *A function $f(x)$ is continuous over an open interval I if $f(x)$ is continuous at every a in the open interval I.*

One can also talk about "left side continuity" and "right side continuity" just like we can talk about "left side limit" and "right side limit." The only difference is that L now equals $f(a)$. The least integer function is not continuous, but it will be from one of the sides. See if you can determine if the greatest integer function is left side continuous or right side continuous.

Example 1. I function can be defined for all real numbers and be continuous at exactly one point and only one point. This example illustrates the difference between a function that is continuous at a point and a function that is continuous on an interval.

Consider a $f(x)$ defined as follows ($\mathbb{Q}$ constitutes the rational numbers and $\mathbb{R}\setminus\mathbb{Q}$ the irrational numbers):

$$f(x) = \begin{cases} |x| & x \text{ is in } \mathbb{Q} \\ -|x| & x \text{ is in } \mathbb{R}\setminus\mathbb{Q} \end{cases}$$

This may be a lot to digest, but if you were 'walking' on the graph of this function, you'd constantly move up and down. You'd be jumping between the graphs. In fact, there'd be no room for your feet to even step because there'd be so many gaps. This is a weird function, so don't think too hard about what it is. Just notice that $\lim_{x\to 0} f(x) = 0 = f(0)$ so the function is continuous at zero. Weird, huh? Yeah. You're welcome :)

2. Intermediate Value Theorem

Functions that are continuous over an interval enjoy a certain property and that property is that the Intermediate Value Theorem holds. The IVT is a theorem that tells you when something exists, but gives you no insight into how to find it. It's like using a metal detector to determine that there is indeed a needle in the haystack, but no indication as to where, exactly. But it's there and knowing the needle is in the haystack means any attempt at finding it will not be in vain.

Theorem 1. *Suppose $f(x)$ is a continuous function over an *closed* interval $[a,b]$ with N a value between $f(a)$ and $f(b)$, with $f(a) \neq f(b)$. Then there exists a value c such that $f(c) = N$.*

This is a very powerful theorem with far reaching consequences. It is frequently used to determine when/if a function crosses the x-axis. It may never be possible to determine a closed formula for calculating the zeroes of a function, but the IVT doesn't care about the explicit value, just that it exists somewhere. In particular, N typically equals zero and the IVT says that there exists some c such that $f(c) = 0$.