Math 2312 Notes
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1. Some derivative rules

Theorem 1 (Sum/difference rule). *If $f(x)$ and $g(x)$ are differentiable functions (meaning we can compute the derivatives of these functions as functions of x), then we have the following rule:*

$$(f \pm g)'(x) = f'(x) \pm g'(x)$$

Proof. We want to prove this. Let's prove that the sum of two differentiable functions is differentiable. Then you can handle the proof of the statement that "The difference of two differentiable functions is differentiable."

Suppose $f(x)$ and $g(x)$ are two differentiable functions. Then this means that $f'(x)$ and $g'(x)$ are well-defined functions (i.e., make sense) for particular values x. We want to resort to the definition to show that $(f+g)'(x)$ makes sense:

$$\begin{aligned}
\lim_{h\to 0} \frac{(f+g)(x+h) - (f+g)(x)}{h} &= \lim_{h\to 0} \frac{f(x+h) + g(x+h) - f(x) - g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h) - f(x) + g(x+h) - g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \qquad (1) \\
&= f'(x) + g'(x).
\end{aligned}$$

□

Note: we used the sum rule for limits to get line (1) of the above calculation. That is, $f'(x)$ and $g'(x)$ exist, so we know that the sum of the limits is the limit of the sum.

Theorem 2 (Constant multiple rule). *If $f(x)$ is a differentiable function and c is some constant, then $(cf)'(x) = cf'(x)$.*

Proof. Let $f(x)$ be a differentiable function and c a constant. We resort to the definition to prove this. Can you finish the following calculation of the limit?

$$
\begin{aligned}
(2) \qquad & \lim_{h\to 0} \frac{(cf)(x+h)-(cf)(x)}{h} = \text{—} \\
(3) \qquad & = \text{—} \\
(4) \qquad & = \text{—} \\
(5) \qquad & = \text{—}
\end{aligned}
$$

□

Question: on what line did you use the constant multiple rule for limits?

2. Derivative of a polynomial

Theorem 3 (Derivative of x^n). *The derivative (as a function of x) of $f(x) = x^n$ is $f'(x) = nx^{n-1}$.*

Proof. Use the binomial theorem and do some algebra. □

Note: the previous theorem assumes n is a positive integer. In reality, the theorem holds for any real number.

Corollary 1 (Derivative of a polynomial). *If $f(x) = a_n x^n + \cdots + a_1 x + a_0$ is a polynomial, then $f'(x) = a_n \cdot nx^{n-1} + \cdots + a_1$.*

Proof. Use the sum rule for derivatives, the constant multiple rule for derivatives and the power rule for derivatives. □