Math 2312 Notes
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1. Limit Laws

We have developed an **intuitive notion of limit** that allows us to analyze the behavior of a function near a point in the plane (really, the behavior of the function when we evaluate the function at values near a number on the x-axis, to be technically correct).

In the lectures to come, we will provide you with a rigorous definition of limit. Why? Everything in mathematics must be on a solid logical foundation. If not, then everything falls apart. In the meantime, let's talk about some limit laws. These laws are facts and can be proven using the rigorous definition of limit that we haven't seen yet. But we can use them, nevertheless.

We have been talking about the limit of a function as the behavior of a function near a point, and this is true. We have seen our intuitive approach to calculating the limit fail us. Let's give ourselves some tools that depart from table-making and lead us somewhere more rigorous.

The limit laws are as follows:

- Sum law
- Difference law
- Constant multiple law
- Product law
- Quotient law
- Power law
- nth-root law

These laws give us a way of examining the behavior of more complicated functions. For example, if $f(x) = x^3$ and $g(x) = 3x - 1$, then what is the limit $\lim_{x \to 1} \dfrac{x^3}{3x-1}$? We can use the limit laws to answer this question, but first we must always check that the limit laws apply! What does this mean? To answer this, we must understand some basic logic. We must understand what it means to make an "if-then" statement.

Suppose we make the statement "If Jose eats his vegetables, then he gets dessert." (Jose is more of a "German Chocolate Cake" kind of guy, in case you're wondering. And always with a large glass of lactose-free milk. No almond milk. Don't you even dare.) If Jose ate his vegetables, but was not given dessert, you would think he'd be justifiably upset that he didn't get his dessert. Now, if he doesn't eat his vegetables, but still is given dessert, then he got it easy! No one is lied to, right? Correct! If he doesn't eat his vegetables and is not provided dessert, then still no one should cry foul, especially Jose. Our "if-then" statements in this section work the same way.

If $\lim_{x \to a} f(x)$ exists and $\lim_{x \to a} g(x)$ exists, then the following statements are true:

- (Sum law) $\lim_{x \to a} f(x) + g(x) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$
- (Difference law) $\lim_{x \to a} f(x) - g(x) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$
- (Constant multiple law) $\lim_{x \to a} cf(x) = c \lim_{x \to a} f(x)$, where c is a constant
- (Product law) $\lim_{x \to a} f(x)g(x) = \lim_{x \to a} f(x) \lim_{x \to a} g(x)$
- (Quotient law) if, in addition to the two limits existing, we have that $\lim_{x \to a} g(x) \neq 0$, then
$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$$
- (Power rule) If, in addition to the two limits existing, we have that n is a positive integer (or a nonzero natural number, if you prefer to say it that way), then $\lim_{x \to a} [f(x)]^n = \left[\lim_{x \to a} f(x)\right]^n$.
- (nth-root rule) This breaks into two cases:
  - (n is odd) If n is odd and a positive integer, then $\lim_{x \to a} [f(x)]^{1/n} = \left[\lim_{x \to a} f(x)\right]^{1/n}$
  - (n is even) If n is even **AND** $\lim_{x \to a} f(x) > 0$, then $\lim_{x \to a} [f(x)]^{1/n} = \left[\lim_{x \to a} f(x)\right]^{1/n}$

One way to think about these extra assumptions in the quotient rule and the nth-root rule is Jose has eaten his vegetables and hasn't disobeyed his parents. If he didn't do anything wrong and also ate his vegetables, then he gets dessert. Now I want cake. Must feed the food-baby. Cake or tacos? Cake or tacos? Both!

Example 1. What is the following limit, if it exists?

$$\lim_{x \to 7} \frac{7x^2 - 3x + 5}{x - 8}$$

Can we use the quotient law? Let's see. Does the limit of the numerator exist? We can use the sum law, difference law, constant multiple law and power law to answer that question in the affirmative. Since $\lim_{x \to 7} x^2$ exists (by the power law), by the constant multiple law, we have that $\lim_{x \to 7} 7x^2 = 7 \lim_{x \to 7} x^2$. Consequently, by the sum and difference laws, we have that

$$\lim_{x \to 7} 7x^2 - 3x + 5 = \lim_{x \to 7} 7x^2 - \lim_{x \to 7} 3x + \lim_{x \to 7} 5. \tag{1}$$

We use the fact that $\lim_{x \to a} x = a$ (something that we can prove rigorously in §2.4) and $\lim_{x \to a} c = c$ to conclude that the limit on line (1) exists.

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\begin{aligned}
\lim_{x\to 7} 7x^2 - \lim_{x\to 7} 3x + \lim_{x\to 7} 5 &= 7\left[\lim_{x\to 7} x\right]^2 - 3\lim_{x\to 7} x + \lim_{x\to 7} 5 \\
&= 7(7)^2 - 3(7) + 5 \\
&= 343 - 21 + 5 \\
&= 327.
\end{aligned}
$$

So, the limit of the numerator exists. Similarly, if you apply the fundamental laws to the denominator, then you will see that the limit of the denominator exists and is also nonzero. So, Jose gets dessert! GO JOSE!

A word or two about two facts: $\lim_{x\to a} x = a$ and $\lim_{x\to a} c = c$. These are two fundamental facts that we can prove. We do not have to accept these as true. Their veracity can be shown. Very few things must be "accepted" as true in math. Those things that are accepted as true are called axioms. Things that follow from from the accepted truth of an axiom are called theorems. They are true, assuming we are working in a logical framework supported by certain axioms. Change the axioms, change the theorems.

But everything we did in the previous example followed from these facts and the limit laws. This is how mathematics works. This is why it can be frustratingly beautiful. It may seem cold at first, but mathematics is a warm art that is waiting for you to hear it speak.

1.1. **Direct Substitution Property.** This sums up the limit laws in the context of polynomials and rational functions. It is a simplified version of a theorem/property that we will soon see. Basically, if $f(x)$ is a polynomial or a rational function and a is in the domain of f, then the limit can be calculated by simply plugging in a into the function and simplifying. Nothing too spectacular here, if you step back and look at it.

2. Squeeze Theorem

Let's make some juice! To wash down the cake, right?

The Squeeze Theorem, also called the Sandwich Theorem or Pinch Theorem, is a statement one can use to calculate the limit of a function, if that function satisfies certain assumptions. Remember Jose? If he ate his vegetables, then he got dessert. Well, if our function eats its vegetables, then we will be able to calculate its limit. Let me explain.

Remember, before we talked about the limit laws, it was imperative that the assumptions be satisfied, namely that each limit exist individually in its own right. We now have a new statement that is in an "if-then" format.

Theorem 1. *If $f(x)$ is a function defined everywhere except possibly at $x = a$, $h(x) \leq f(x) \leq g(x)$ for all values except possibly $x = a$ and $c = \lim_{x\to a} h(x) = \lim_{x\to a} g(x)$, then $\lim_{x\to a} f(x)$ exists and is equal to c.*

Notice the three things that must be true for the "if-then" statement to work. Since this is a theorem, if the three things in the "if" part are true, it must be that the limit exists and is equal to c.

Consider the following example.

Example 2. Let $f(x) = x\sin(1/x)$. This function is not defined at $x = 0$, so we cannot simply plug zero into the function. Moreover, the limit of $\sin(1/x)$ as x gets closer and closer to zero does not exist. So we cannot use our limit laws. If the limit exists, it's because of some other reason. Think about Jose still getting dessert even if he doesn't eat his vegetables. The statement isn't wrong, we just can't conclude anything about him getting dessert or not. Moreover, if $\lim_{x\to 0} x\sin(1/x)$ does not exist, then are we surprised? We wouldn't be, because $\lim_{x\to 0}\sin(1/x)$ does not exist.

But we will use the Squeeze Theorem to show that $\lim_{x\to 0} x\sin(1/x)$ does in fact exist. We just can't use the limit laws to show this. In other words, maybe Jose got an A+ on his Calculus exam and his parents decided on a new rule: "If you get an A+ on your Calculus exam, then you get dessert." The Squeeze Theorem is that new rule.

Now, $f(x)$ is not defined at zero, but the Squeeze Theorem doesn't care if it is. First, recognize that $-1 \leq \sin(1/x) \leq 1$, since sin is a bounded function (think about the values of sin and it's graph). The next few sentences are tricky, so write them down as you read. Multiply through the inequality by x and assume this variable takes on positive values. You should get $-x \leq x\sin(1/x) \leq x$. Now assume $x < 0$. Then the inequalities all flip but $-x > 0$. Consequently, $-x \geq x\sin(1/x) \geq x$. Therefore, $x\sin(1/x)$ squeezed between $-|x|$ and $|x|$:

$$-|x| \leq x\sin(1/x) \leq |x| \tag{2}$$

for all $x \neq 0$. Since $\lim_{x\to 0}|x| = \lim_{x\to 0} -|x| = 0$, the Squeeze Theorem tells us that the limit of $f(x)$ as x goes to zero must be equal to zero, as well.